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MAXIMALITY IN MODAL LOGIC

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0. Introduction

Let (FP) be a formal system for a version of propositional logic. For example, (FP) might be a formalization of intuitionistic, classical or some form of modal propositional logic. Let (FA) be the corresponding system of first-order arithmetic. We say (FP) is *maximal for* (FA) if for any formula $A(p_1, \ldots, p_m)$ of (FP), if $\nvdash_{\mathrm{FP}} A$, then there are sentences $B_1, \ldots, B_m$ of (FA) such that $\nvdash_{\mathrm{FA}} A(B_1, \ldots, B_m)$.

Maximality for classical and intuitionistic systems has been studied by several authors. For a detailed discussion, we refer the reader to Leivant [6]. In the present paper we consider the problem of maximality for two systems of modal logic. The first system, which we denote by (GrzP), was introduced by Grzegorczyk [4]. Recent interest in (GrzP) is due to the role it plays in the logic of provability (cf. Boolos [1], [2]). The second system, denoted by (EP) for 'epistemic propositional logic', is Lewis' system (S4). Epistemic formal systems have assumed considerable importance because they provide a setting for integrating classical and intuitionistic mathematics. The book Shapiro [11] contains several papers on this topic, see also [3] and [8].

Our proof of the maximality of (GrzP) for (GrzA), which is quite short, is modelled directly on Smorynski's argument [12] for the corresponding result for intuitionistic logic. This is possible because the Kripke semantics for (GrzP) is completely analogous to that of intuitionistic propositional logic. In contrast, our proof of the maximality of (EP) for (EA), the main ideas of which is due to the second author, is rather involved. The main complication here results from the form of the completeness theorem for (EP), which requires infinite trees. The problem of building Kripke models for (EA) over infinite trees led to the consideration of a combinatorial principle, which roughly amounts to an iterated version of the finite Ramsey Theorem.

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1. Formal systems

Our two-systems of modal propositional logic, (EP) and (GrzP), are based on the same primitive logical constants: $\perp$, $\rightarrow$ and $\Box$. Formulas are built up in the usual way. We will use $p, q, r, \ldots$ as syntactic variables for propositional letters. As syntactic variables for formulas, the letters $A, B, C, \ldots$ will be used.

We will write

$$\begin{aligned} \neg A \quad &\text{for} \quad (A \rightarrow \perp), \\ (A \vee B) \quad &\text{for} \quad (\neg A \rightarrow B), \\ (A \vee B) \quad &\text{for} \quad \neg(A \rightarrow \neg B), \quad \text{and} \\ (\Diamond A) \quad &\text{for} \quad \neg\Box\neg A. \end{aligned}$$

(EP) has as axioms all tautologies and all sentences $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$, $\Box A \rightarrow A$, $\Box A \rightarrow \Box\,\Box A$; its rules are modus ponens:

$$\frac{A \qquad A \rightarrow B}{B}$$

and necessitation:

$$\frac{A}{\Box A}.$$

(GrzP) is obtained from (EP) by adding as axioms all sentences

$$\Box(\Box(A \rightarrow \Box A) \rightarrow A) \rightarrow A.$$

As descriptive constants for arithmetic, we take the usual ones: 0, $(\ )'$, $+$, $\cdot$ and $<$. The systems (EA) and (GrzA) of modal arithmetic are obtained from the corresponding propositional systems by adding:

(1) Axioms for equality:

$$x = x, \qquad x = y \rightarrow (A(x) \rightarrow A(y)).$$

(2) Axioms and a rule for the universal quantifier:

$$\forall x\,(A \rightarrow B) \rightarrow (A \rightarrow \forall x\, B), \quad x \text{ not free in } A,$$

$$\forall x\, A(x) \rightarrow A(t), \qquad \frac{A}{\forall x\, A}.$$

(3) The arithmetic axioms: the usual axioms for 0 and $(\ )'$, the recursive defining axioms for $+$, $\cdot$ and $<$, and the scheme of induction for *all* formulas of the language.

If (S) is one of the systems (EP), (GrzP), (EA) or (GrzA), and A is a formula of (S), we will write $\vdash_S A$ to indicate that A is provable in (S).

2. Kripke semantics

Completeness theorems for (EP) and (GrzP) for Kripke Models have been given by Kripke [5] and Segerberg [10], respectively. Their formulations are not quite adequate for the applications we have in mind. This section is devoted to refining these completeness theorems for present purposes. For this we use ideas from Schumm [9] and Smorynski [12].

2.1. Definition. A *Kripke model* $\mathcal{K}$ consists of a poset $\langle P, \leqslant\rangle$ and a *forcing relation* $\Vdash$ between elements of P and propositional variables.

Let $\mathcal{K} = \langle P, \leqslant, \Vdash\rangle$ be a Kripke model. We will write $\Vdash_a^{\mathcal{K}} P$ if $\langle a, p\rangle \in \Vdash$ and $\nVdash_a^{\mathcal{K}} P$ otherwise. If there is no chance of confusion, we will omit the superscript $(\ )^{\mathcal{K}}$. The relation $\Vdash_a A$ where A is an arbitrary modal formula and $a \in P$ is defined inductively by the classes:

(1) $\nVdash_a \bot$;

(2) $\Vdash_a (A \rightarrow B)$ iff ($\Vdash_a A$ implies $\Vdash_a B$);

(3) $\Vdash_a \Box A$ iff for all $b \geqslant a$ $\Vdash_b A$.

We say $\mathcal{K}$ *satisfies* A, denoted by $\mathcal{K} \Vdash A$, if for all $a \in P$, $\Vdash_a A$.

2.2. Definition. Let $\mathbb{K}$ be a class of Kripke Models. We say (GrzP) (resp., (EP)) is *complete for* $\mathbb{K}$ if for each modal formula A, $\vdash_{\mathrm{GrzP}} A$ (resp. $\vdash_{\mathrm{EP}} A$) if and only if for every $\mathcal{K}$ in $\mathbb{K}$, $\mathcal{K} \Vdash A$.

A Kripke model $\mathcal{K} = \langle P, \leqslant, \Vdash\rangle$ is called *finite* if P is a finite set.

2.3. Theorem (Segerberg [10]). (GrzP) *is complete for the class of finite Kripke models.*

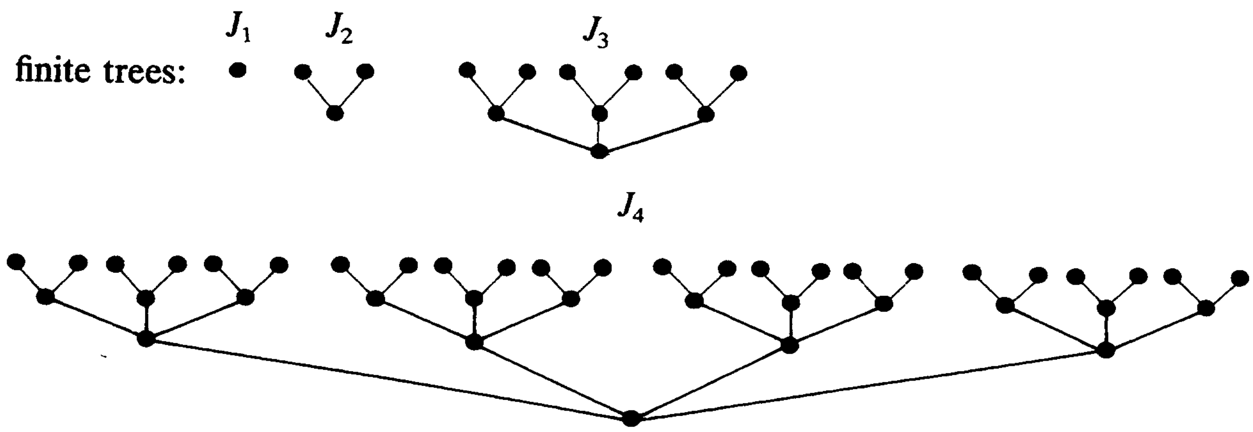
We wish to show that (GrzP) is complete for the class of Kripke models whose underlying poset is a special type of finite tree. Our argument is a simple modification of Smorynski's method [12].

In the sequel, the small Greek letters $\alpha, \beta, \gamma, \ldots$ will denote finite sequences of natural numbers. We will write $\alpha \leqslant \beta$ if α is an initial segment of β. $\alpha * \beta$ denotes the sequence obtained by concatenating α and β. Finally, αn denotes $\alpha * \langle n\rangle$.

By a *tree* we understand a nonempty set of finite sequences of natural numbers closed under taking initial segments.

A Kripke model $\mathcal{K} = \langle P, \leqslant, \Vdash\rangle$ is called a *tree model* if P is a tree and $\leqslant$ is the standard partial ordering $\leqslant$ on P.

2.4. Definition. The *modified Jaskowski sequence* is the following sequence of finite trees:



2.5. Lemma. *Let $(P, \leqslant)$ be a finite poset and let $a \in P$. Then there is a Jaskowski tree J_n and a mapping $\phi : \{b \in P; b \geqslant a\} \to J_n$ satisfying the following*:

(i) *For all b, $b' \geqslant a$, $b \leqslant b'$ if and only if $\phi(b) \leqslant \phi(b')$.*

(ii) $\phi(a) = \langle\ \rangle$.

(iii) *For all b, $b' \geqslant a$, b' is an immediate succesor of b iff $\phi(b')$ is an immediate successor of $\phi(b)$.*

A Kripke model is called a *Jaskowski model* if its underlying poset is a Jaskowski tree.

2.6. Completeness Theorem for (GrzP). (GrzP) *is complete for the class of Jaskowski Kripke models.*

Proof. Suppose $\nvdash_{\mathrm{GrzP}} A$. Let $\mathcal{K} = \langle P, \leqslant, \Vdash\rangle$ be such that $\mathcal{K} \not\Vdash A$ and P is finite. Let $a \in P$ satisfy $\not\Vdash_a A$. We can evidently assume that a is the initial node of P and by Lemma 2.5 we may assume that P is a subtree of a Jaskowski tree J_n with $a = \langle\ \rangle$.

Claim. There is a forcing relation $\Vdash^*$ on J_n such that for any modal formula B and any $\alpha \in P$

$$\Vdash_\alpha B \quad \text{iff} \quad \Vdash^*_\alpha B.$$

For each $\alpha \in P$, define

$$\Vdash^*_\alpha P \quad \text{iff} \quad \Vdash_\alpha P.$$

For each $\beta \in P$, choose a terminal node $t_\beta \geqslant \beta$ in P. Let $\alpha \in J_n - P$, let $\beta \in P$ be maximal such that $\beta \leqslant \alpha$. Then define

$$\Vdash^*_\alpha P \quad \text{iff} \quad \Vdash^*_{t_\beta} P.$$

By an easy induction on the complexity of formulas we have

(1) For $\alpha \in P$, $\Vdash_\alpha B$ iff $\Vdash^*_\alpha B$.

(2) For $\alpha \in J_n - P$, $\Vdash^*_\alpha B$ iff $\Vdash_{t_\beta} B$, for all formulas B.

From the claim it follows that $\not\Vdash^*_{\langle\ \rangle} A$ and so $\langle J_n, \leqslant, \Vdash^*\rangle \not\Vdash A$. □

To obtain the appropriate completeness theorem (EP), we modify Schumm's argument [9]. Let A be a formula of (EP) such that $\not\Vdash_{\mathrm{EP}} A$. Let Γ be the set of all formulas of the form B or $\neg B$, where B is a subformula of A. Fix some enumeration of the maximal consistent subsets of Γ. For Δ a consistent subset of Γ, we will write Δ^+ for the first maximal consistent subset of Γ containing Δ. Finally, for any $\Delta \subseteq \Gamma$ and any formula C such that $(\neg\Box C) \in \Delta$ define

$$\Delta(C) = \{\neg C\} \cup \{\Box B; \Box B \in \Delta).$$

Now suppose $\Box A_0, \ldots, \Box A_{k-1}$ are all the subformulas of A beginning with $\Box$. We assign maximal consistent subsets Δ_α to each node $\alpha \in \omega^{<k}$ as follows:

$$\Delta_{\langle\ \rangle} = \{\neg A\}^+,$$

$$\Delta_{\alpha n} = \begin{cases} \Delta_\alpha(A_n)^+ & \text{if } (\neg\Box A_n) \in \Delta, \\ \Delta_\alpha & \text{otherwise.} \end{cases}$$

Define a forcing relation on the tree $\omega^{<k}$ by

$$\Vdash_\alpha P \quad \text{iff} \quad p \in \Delta_\alpha.$$

2.7. Lemma. *For each subformula B of A,*

$$\Vdash_\alpha B \quad \textit{iff} \quad B \in \Delta_\alpha.$$

Proof. We argue by induction on the complexity of B. The interesting case is when $B \equiv \Box C$. Then

$$\begin{aligned} \Vdash_\alpha B \quad &\Leftrightarrow \quad \forall \beta \geqslant \alpha \Vdash_\beta C \\ &\Leftrightarrow \quad \forall \beta \geqslant \alpha\ C \in \Delta_\beta. \end{aligned}$$

If $(\Box C) \in \Delta_\alpha$, then evidently $(\Box C) \in \Delta_\beta$ for all $\beta \geqslant \alpha$. Thus $C \in \Delta_\beta$ for all $\beta \geqslant \alpha$, and so $\Vdash_\alpha$B. Conversely, if $(\Box C) \notin \Delta_\alpha$, then $(\neg\Box C) \in \Delta_\alpha$ and $(\neg C) \in \Delta_\beta$ for some $b \geqslant \alpha$. We get $\Vdash_\alpha B$. $\Box$

It follows at once that $\not\Vdash_{\langle\ \rangle} A$. So (EP) is complete for finitely branching Kripke models. However, the model just constructed has a special property beyond being finitely branching. To formulate this, we will need the following definition.

2.8. Definition. Let $\mathcal{K} = \langle T, \Vdash \rangle$ be a tree model.

(1) For each node $\alpha \in T$, let $\mathcal{K}_\alpha = \langle T^{(\alpha)}, \Vdash^{(\alpha)} \rangle$ be the Kripke model defined by

$$T^{(\alpha)} = \{\beta; \alpha * \beta \in T\}, \qquad \Vdash_\alpha^{(\alpha)} P \quad \Leftrightarrow \quad \Vdash_{\alpha*\beta} P.$$

(2) Two nodes α, $\beta \in T$ are said to be equivalent, denoted by $\alpha \equiv_{\mathcal{K}} \beta$, if $\mathcal{K}_\alpha = \mathcal{K}_\beta$.

2.9. Definition. A tree model $\mathcal{K} = \langle T, \Vdash \rangle$ is said to *have finite type* if there are only finitely many equivalence classes under the relation $\equiv_{\mathcal{K}}$.

2.10. Completeness Theorem for (EP). (EP) *is complete for the class of tree models of finite type.*

We close this section by recording the Soundness Theorems for the obvious extension of Kripke Semantics to systems of Arithmetic.

2.11. Definition. A *quantificational Kripke model* consists of a poset $\langle P, \leqslant \rangle$ and an assignment $\mathcal{A}_a = \langle A_a, 0, (\ )', +, \cdot, < \rangle$ of a first-order structure, of the proper type to interpret the language of arithmetic, to each element a of P such that for $a \leqslant b$ in P, $\mathcal{A}_a$ is a submodel of $\mathcal{A}_b$.

Again we call a quantificational Kripke model *finite* if the underlying poset is finite.

2.12. Definition. Let $\mathcal{K} = \langle P, \leqslant, \mathcal{A} \rangle$ be a quantificational Kripke model. The *forcing relation* $\Vdash_p A[a_1, \ldots, a_n]$, where A is a formula of modal arithmetic, all the free variables of which are along $x_1, \ldots, x_n$, $p \in P$ and $a_1, \ldots, a_n \in \mathcal{A}_p$, is defined by induction on the complexity of A as follows:

(1) For A atomic, $\Vdash_p A[\boldsymbol{a}]$ iff $\mathcal{A}_p \vDash A[\boldsymbol{a}]$.
(2) $\Vdash_p (A \rightarrow B)[\boldsymbol{a}]$ iff ($\Vdash_p A[\boldsymbol{a}]$ implies $\Vdash_p B[\boldsymbol{a}]$).
(3) $\Vdash_p \forall y\, A[\boldsymbol{a}]$ iff $\forall b \in \mathcal{A}_p \Vdash_p A[b\boldsymbol{a}]$.
(4) $\Vdash_p \Box A[\boldsymbol{a}]$ iff $\forall q \geqslant p \Vdash_q A[\boldsymbol{a}]$.

We will say that a quantificational Kripke model $\mathcal{K}$ *satisfies the modal sentence* A, denoted by $\mathcal{K} \Vdash A$, if for all $p \in P$, $\Vdash_p A$.

2.13. Theorem. *Let $\mathcal{K}$ be a quantificational Kripke model satisfying all the arithmetical axioms. Then*

(1) *Any theorem of* (EA) *is satisfied in $\mathcal{K}$.*
(2) *If $\mathcal{K}$ is finite, then any theorem of* (GrzA) *is satisfied in $\mathcal{K}$.*

3. Maximality for (Grz)

Since the Kriple semantics for (GrzP) is completely parallel to Kripke semantics for intuitionistic propositional logic, it is possible to translate Smorynski's proof of the maximality of intuitionistic propositional logic for intuitionistic arithmetic to the modal setting and obtain the maximality of (GrzP) for (GrzA).

Suppose $A(p_1, \ldots, p_m)$ is a formula of (GrzP) such that

$$\nvdash_{\mathrm{GrzP}} A.$$

By the completeness theorem for (GrzP), let $\mathcal{K} = \langle J_n, \Vdash \rangle$ be a Kriple model over the Jaskowski tree J_n such that

$$\not\Vdash^{\mathcal{K}}_{\langle\ \rangle} A.$$

Choose $B_1, \ldots, B_{n!}$ Σ^0_1-sentences of arithmetic completely independent over (PA). For each q, $1 \leqslant q \leqslant n!$, let $\mathscr{A}_q$ be a model of (PA) such that

$$\mathscr{A}_q \Vdash B_p \quad \Leftrightarrow \quad p = q$$

for $1 \leqslant p \leqslant n!$

Let $\alpha_1, \ldots, \alpha_{n!}$ be the top nodes of J_n. Define $\bar{\mathcal{K}}$ to be the quantificational Kripke model over J_n obtained by assigning the model $\mathscr{A}_q$ to the node α_q, for $1 \leqslant q \leqslant n!$, and assigning the standard model $\mathcal{N}$ to all the non-terminal nodes of J_n.

3.1. Lemma. *$\bar{\mathcal{K}}$ is a model of* (GrzA).

3.2. Definition. (1) For each node α of J_n let B_α be the formula

$$\bigwedge_{q:\alpha_q \geqslant \alpha} \Diamond B_q \wedge \bigwedge_{q:\alpha_q \not\geqslant \alpha} \Box\neg B_q.$$

(2) For each subset Γ of J_n let B_Γ be the formula

$$\bigvee_{\alpha \in \Gamma} B_\alpha.$$

3.3. Lemma. (1) *For each pair of nodes α, β of J_n*

$$\Vdash^{\bar{\mathcal{K}}}_{\alpha} B_\beta \quad \Leftrightarrow \quad \alpha = \beta.$$

(2) *For each subset Γ of J_n and each node α*

$$\Vdash^{\bar{\mathcal{K}}}_{\alpha} B_\Gamma \quad \Leftrightarrow \quad \alpha \in \Gamma.$$

Now let $\Gamma_i = \{\alpha \in J_n; \Vdash^{\mathcal{K}}_{\alpha} P_i\}$, for $1 \leqslant i \leqslant m$. And let $C_i = B_{\Gamma_i}$. By induction on the complexity of formulas we easily obtain

3.4. Lemma. *For each formula $D(p_1, \ldots, p_m)$ of* (GrzP) *containing only the propositional variables shown*

$$\Vdash^{\mathcal{K}}_{\alpha} D(p_1, \ldots, p_m) \quad \Leftrightarrow \quad \Vdash^{\bar{\mathcal{K}}}_{\alpha} D(C_1, \ldots, C_m),$$

for any node α of J_n.

3.5. Theorem. *Suppose $A(p_1, \ldots, p_m)$ is a formula of* (GrzP) *such that*

$$\not\Vdash_{\mathrm{GrzP}} A.$$

Then there are sentences $B_1, \ldots, B_m$ of (GrzA) *such that*

$$\not\Vdash_{\mathrm{GrzP}} A(B_1, \ldots, B_m).$$

4. Iterating partition properties

In Section 5 we will need to show the consistency of a theory extending first-order arithmetic which contains a tower of 'Paris–Harrington indiscernibles' [7]. For this purpose we use a combinatorial principle which, roughly, amounts to an iterated version of the Finite Ramsey Theorem. Before stating the principle, we will need to introduce some terminology.

Let k and $l_1, \ldots, l_n$ be natural numbers. By a relation of *type* $(k; \boldsymbol{l})$ we understand a subset of

$$\omega^k \times [\omega]^{l_1} \times \cdots \times [\omega]^{l_n},$$

where, as usual, $[\omega]^l$ denotes the set of increasing sequences $a_0 < a_1 < \cdots < a_{l-1}$, from ω.

4.1. Definition. (1) Let R be a relation of type $(k; l)$. A subset X of ω is *$*$-homogeneous for* R if for all sequences $\boldsymbol{a}$, $\boldsymbol{b} \in [X]^l$, all $c \in X$ such that $c < \min\{a_0, b_0\}$, we have

$$\forall z_0, \ldots, z_{k-1} < c\, (R(\boldsymbol{z}; \boldsymbol{a}) \quad \Leftrightarrow \quad R(\boldsymbol{z}; \boldsymbol{b})).$$

(2) Let $F : [\omega]^l \to r$ be a partition of $[\omega]^l$ into r pieces. A subset X of ω^l is *homogeneous for* F if for all $\boldsymbol{a}$, $\boldsymbol{b} \in [X]^l$, we have

$$F(\boldsymbol{a}) = F(\boldsymbol{b}).$$

4.2. Definition. A family of relations $R_1, \ldots, R_n$, where R_i has type $(k_i; l_i)$, is said to be *bounded by* N if

$$N > \mathrm{Max}\{n, k_1, \ldots, k_n, l_1, \ldots, l_n\}.$$

It will be convenient below to regard any number-theoretic relation with $(k + \sum_{i=1}^{n} l_i)$ arguments as a relation of type $(k; \boldsymbol{l})$.

4.3. Definition. The relation $X \xrightarrow[t]{} (N)^e_r$, where N, t, e, r are natural numbers and $X \subseteq \omega$, is defined by induction on t as follows:

(1) $X \xrightarrow[1]{} (N)^e_r$ if the cardinality of X is greater than N.

(2) $X \xrightarrow[t+1]{} (N)^e_r$ if for any family of partitions $F_1, \ldots, F_N : [X]^e \to r$ and any family of relations $R_1, \ldots, R_n$ bounded by N, there exists a subset Y of X such that Y is homogeneous for $F_1, \ldots, F_n$, $*$-homogeneous for $R_1, \ldots, R_n$ and $Y \xrightarrow[t]{} (N)^e_r$.

4.4. Theorem. *For all N, t, e, r and any infinite set B, there exists a finite subset X of B such that*

$$X \xrightarrow[t]{} (N)^e_r.$$

Proof. Let N, e, r and B be given. We argue by induction on t. The case $t = 1$ is clear. Suppose the result fails for $t+1$. By a counterexample for M, where M is a natural number, we understand a sequence of relations $R_1, \ldots, R_n$, where $R_i \subseteq M^{k_i} \times [M]^{l_i}$, bounded by N and a sequence of partitions $F_1, \ldots, F_n : [M]^e \to r$ such that there exists no subset $Y \subseteq M \cap B$ which is $*$-homogeneous for $R_1, \ldots, R_n$, homogeneous for $F_1, \ldots, F_N$ and satisfies $Y \underset{t}{\to} (N)^e_r$. These counterexamples naturally form a finitely branching infinite tree. By König's Lemma there is a sequence of relations $R_1, \ldots, R_n$, bounded by N, and a sequence of partitions $F_1, \ldots, F_N : [w]^e \to r$ such that for each M the restrictions of R_i to $M^{k_i} \times [M]^{l_i}$, for $1 \leqslant i \leqslant n$, and F_i to $[M]^e$, for $1 \leqslant i \leqslant N$, form a counterexample for M. Now define partitions $S_i : [w]^{2l_i+1} \to 2$, $1 \leqslant i \leqslant n$, by the prescription

$$S_i(c, \boldsymbol{a}, \boldsymbol{b}) = \begin{cases} 0 & \text{if for all } z < c \ (R_i(z; \boldsymbol{a}) \quad \Leftrightarrow \quad R_i(z; \boldsymbol{b})), \\ 1 & \text{otherwise.} \end{cases}$$

Since we can combine the partitions $S_1, \ldots, S_n$, $F_1, \ldots, F_N$ into a single partition, by the Infinite Ramsey Theorem, there exists an infinite subset H of B which is simultaneously homogeneous for $S_1, \ldots, S_n$, $F_1, \ldots, F_N$. Now, by the induction hypothesis, let X be a finite subset of H such that $X \underset{t}{\to} (N)^e_r$. But then by taking $M = \max(X) + 1$, we get a homogeneous subset X of $M \cap B$ satisfying $X \underset{t}{\to} (N)^e_r$, which is absurd. □

Below we will need the fact that for fixed N, t, e and r, there is a version of Theorem 4.4 which is provable in (PA). To formulate things in a first-order way we will code finite sets by integers, replace B by a formula and interpret 'infinite' to mean unbounded. This results in a scheme;

$$\mathrm{IRT}(N, t, e, r): \quad \begin{array}{l} \forall x\ \exists y > x\ A(y) \to \exists X \\ (\text{“}X \text{ is a finite set”} \ \wedge \forall x \in X\ A(X) \wedge X \underset{t}{\to} (N)^e_n). \end{array}$$

4.5. Corollary. *For each N, t, e and r, the scheme* IRT(N, t, e, r) *is provable in* (PA).

Proof. This follows from the fact that for fixed e, the Infinite Ramsey Theorem $\omega \to (\omega)^e_r$ is provable in (PA) and the fact that a definable version of König's Lemma (taking left most branches) is also provable in (PA). □

We should also note that for fixed t, the relation $R(X, N, e, r)$ defined by

$$R(X, N, e, r) \quad \Leftrightarrow \quad X \underset{t}{\to} (N)^e_r,$$

is definable by a Σ^0_1-formula. This follows by a simple induction on t.

4.6. Definition. Let $\mathscr{R}$ be a family of relations

$$R_{11}, \ldots, R_{1m_1}, \ldots, R_{t1}, \ldots, R_{tm_t},$$

where R_{ij} has type $(k_{ij}; l_{ij}^{(1)}, \ldots, l_{ij}^{(i)})$. The *n-rank* of $\mathfrak{R}$ is the pair (t, N), where

$$N = \mathrm{Max}\left\{\sum_{i=1}^{t} n^{i-1}m_i,\ k_{ij} + \sum_{s=1}^{i} l_{ij}^{(s)}\right\}.$$

The peculiar choice of N is dictated by the inuctive argument used to prove Lemma 4.8.

4.7. Definition. Let $\mathfrak{R}$ be a family of relations. A sequence of finite subsets $X_1, X_2, \ldots$ of natural numbers is *n-homogeneous for* $\mathfrak{R}$ if the following conditions are satisfied:

(1) If $i_0 < i_1$, $x \in X_{i_0}$ and $y \in X_{i_1}$, then $x < y$.

(2) For each relation $R \in \mathfrak{R}$, where R has type $(k; l^{(1)}, \ldots, l^{(s)})$, each pair of sequences $i_1 < \cdots < i_s$ and $i_1' < \cdots < i_s'$, satisfying $i_1 = i_1'$ and for $1 < q \leqslant s$, $i_q \equiv i_q' \pmod n$, each family of sequences $y_1 \in [X_{i_1}]^{l^{(1)}}, \ldots, y_s \in [X_{i_s}]^{l^{(s)}}$ and $y_1' \in [X_{i_1'}]^{l^{(1)}}, \ldots, y_s' \in [X_{i_s'}]^{l^{(s)}}$ and each $y \in X_{i_1}$ with $y < \min\{y_{10}, y_{10}'\}$ we have

$$\forall z < y\ (R(z; y_1, \ldots, y_s) \iff R(z; y_1', \ldots, y_s)).$$

4.8. Lemma. *Let $\mathfrak{R}$ be a family of relations of rank (t, N). Let $X_1, X_2, \ldots$ be a sequence of finite subsets of ω satisfying the following conditions*:

(1) *If $i < i'$, $x \in X_i$ and $y \in X_{i'}$, then $x < y$.*

(2) *For each i*

$$X_i \xrightarrow[t+1]{} (M)_{r_i}^{M},$$

where $M > N$ and r_i is defined by

$$r_0 = 0, \qquad r_{i+1} = 2^{(\sup X_i)^N}.$$

Then there exists a subsequence $X_{\pi(1)}, X_{\pi(2)}, \ldots$ of $X_1, X_2, \ldots$ and subsets $X_1' \subseteq X_{\pi(1)}$, $X_2' \subseteq X_{\pi(2)}, \ldots$ such that the following conditions are satisfied:

1) *The sequence $X_1', X_2', \ldots$ is n-homogeneous for $\mathfrak{R}$.*

2) *For each i, $\pi(i) \equiv i \pmod n$.*

3) *For each i, $\|X_i'\| > M$.*

Proof. If $t = 1$, so that $\mathfrak{R}$ is a family $R_1, \ldots, R_m$, where R_i is of type $(k_i; l_i)$ and $M > m$, k_i, l_i, we may choose $X_i' \subseteq X_i$ $*$-homogeneous for $R_1, \ldots, R_m$ such that $\|X_i'\| > M$.

Suppose $t = \bar{t} + 1$. Then $\mathfrak{R} = \bar{\mathfrak{R}} \cup \{R_1, \ldots, R_m\}$, where R_p has type $(k_p; l_p^{(1)}, \ldots, l_p^{(t)})$ and $\bar{\mathfrak{R}}$ has rank $(\bar{t}, N')$ for some N'. For each of the relations R_p let

$$F_p : [w]^{l_p^{(t)}} \to \mathcal{P}(\omega^{k_p + \sum_{j=1}^{\bar{t}} l_p^{(j)}})$$

be the function defined by

$$(z; y_1, \ldots, y_{\bar{t}}) \in F_p(y) \iff R_p(z; y_1, \ldots, y_{\bar{t}}, y).$$

By restriction, we obtain a partition

$$F_p : [X_{i+1}]^{l_p^{(i)}} \to 2^{(\sup X_i)^K} \quad \text{where } K = \left(k_p + \sum_{j=1}^{\bar{i}} l_p^{(j)}\right).$$

By our choice of r_{i+1} and the definition of N, we may choose $Y_{j+1} \subseteq X_{i+1}$ satisfying Y_{i+1} is homogeneous for the partitions $F_1, \ldots, F_m$ and

$$Y_{i+1} \xrightarrow[\bar{i}+1]{} (M)^M_{r_{i+1}}.$$

Let $Y_1 = X_1$.

The sequence $Y_1, Y_2, \ldots$ has the property that for all $i < i'$ and all y, $y' \in [Y_{i'}]^{l_p^{(i)}}$

$$F_p(y) \mid Y_i = F_p(y') \mid Y_i, \quad 1 \leqslant p \leqslant m.$$

(By $F_p(y) \mid Y_i$ we mean the relation of $k_p + \sum_{j=1}^{\bar{i}} l_p^{(j)}$ arguments determined by the restriction of $F_p(y)$ to the set $\{n; n \leqslant \sup Y_i\}$.) Using this property, we will now define, inductively, a doubly indexed sequence

$$\begin{array}{ll} Y_1^{(0)}, & Y_2^{(0)}, \ldots \\ Y_1^{(1)}, & Y_2^{(1)}, \ldots \\ \vdots & \vdots \end{array}$$

having the following properties:

(1) For each j, $Y_1^{(j)}, Y_2^{(j)}, \ldots$ is a subsequence of $Y_1, Y_2, \ldots$ such that for each i, $\pi(i) \equiv i \pmod n$.

(2) If $i < q \leqslant j+1$, then for any q' with $q \equiv q' \pmod n$, any $y \in [Y_q^{(j)}]^{l_p^{(i)}}$, $y' \in [Y_{q'}^{(j)}]^{l_p^{(i)}}$ we have

$$F_p(y) \mid Y_i^{(j)} = F_p(y') \mid Y_i^{(j)}, \quad 1 \leqslant p \leqslant m.$$

Let $Y_i^{(0)} = Y_i$. The construction of the sequence $Y_1^{(j+1)}, Y_2^{(j+1)}, \ldots$ from $Y_1^{(j)}, Y_2^{(j)}, \ldots$ uses a simple pigeon hole argument.

Let $Y_{\sigma(1)}^{(j)}, Y_{\sigma(2)}^{(j)}, \ldots$ be a subsequence of $Y_{j+2}^{(j)}, Y_{j+3}^{(j)}, \ldots$ such that for each i, $\sigma(i) \equiv j+2 \pmod n$ and for each i, i', $y \in [Y_{\sigma(i)}^{(j)}]^{l_p^{(i)}}$, $y' \in [Y_{\sigma(i')}^{(j)}]^{l_p^{(i)}}$, we have

$$F_p(y) \mid Y_{j+1}^{(j)} = F_p(y') \mid Y_{j+1}^{(j)}, \quad 1 \leqslant p \leqslant m.$$

Now let $Y_1^{(j+1)}, Y_2^{(j+1)}, \ldots$ be the sequence

$$Y_1^{(j)}, Y_2^{(j)}, \ldots, Y_{j+1}^{(j)}, Y_{\sigma(1)}^{(j)}, Y_{\sigma(1)+1}^{(j)}, \ldots, Y_{\sigma(1)+(n-1)}^{(j)},$$
$$Y_{\sigma(2)}^{(j)}, Y_{\sigma(2)+1}^{(j)}, \ldots, Y_{\sigma(2)+(n-1)}^{(j)}, \ldots$$

This new sequence clearly satisfies the requirements above. Now define $Y_i^{(\infty)} = Y_i^{(i)}$. Then $Y_i^{(\infty)} \subseteq X_{\pi(i)}$, where $\pi(i) \equiv i \pmod n$ and for all i, i', j such that $i \equiv i'$ $\pmod n$ and $j < \min\{i, i'\}$ we have

$$F_p(y) \mid Y_j^{(\infty)} = F_p(y') \mid Y_j^{(\infty)}, \quad 1 \leqslant p \leqslant m$$

for all $y \in [Y_i^{(\infty)}]^{l_p^{(j)}}$, $y' \in [Y_{i'}^{(\infty)}]^{l_p^{(j)}}$.

Define new relations $R_p^{(q)}$ of type $(k_p; l_p^{(1)}, \ldots, l_p^{(\bar{t})})$, for $1 \leqslant q \leqslant n$ and $1 \leqslant p \leqslant m$, by the prescription

$$R_p^{(q)}(z; y^{(1)}, \ldots, y^{(\bar{t})})$$

iff for some (any) $i \equiv q \pmod n$ with $z, y^{(1)}, \ldots, y^{(\bar{t})} < \sup Y_{i-1}^{(\infty)}$, for some (any) $y \in [Y_i^{(\infty)}]^{l_p^{(t)}}$ we have

$$(z; y^{(1)}, \ldots, y^{(\bar{t})}) \in F_p(y).$$

Let $\bar{\mathcal{R}} = \mathcal{R}' \cup \{R_p^{(q)}\}_{1 \leqslant q \leqslant n, 1 \leqslant p \leqslant m}$. Then $\bar{\mathcal{R}}$ has rank $(\bar{t}, \bar{N})$, where $\bar{N} \leqslant N$. But evidently

$$Y_i^{(\infty)} \xrightarrow[\bar{t}]{} (M)_{r_i}^M$$

with $M > \bar{N}$ and $r_{i+1} \geqslant 2^{(\sup Y_i^{(\infty)})^N}$. Consequently by the induction hypothesis, we may choose a subsequence $Y_{\pi(1)}^{(\infty)}, Y_{\pi(2)}^{(\infty)}, \ldots$ of $Y_1^{(\infty)}, Y_2^{(\infty)}, \ldots$ and subsets $X_i' \subseteq Y_{\pi(i)}^{(\infty)}$ such that

(1) The sequence $X_1', X_2', \ldots$ is n-homogeneous for $\bar{R}$.

(2) For each i, $\pi(i) \equiv i \pmod n$.

(3) For each i, $\|X_i'\| > M$.

We claim that the sequence $X_1, X_2, \ldots$ is actually n-homogeneous for $\mathcal{R}$. We need only check those $R \in \mathcal{R} - \mathcal{R}'$. Let R of type $(k; l^{(1)}, \ldots, l^{(t)})$ be one of these. Also let F be the function

$$[\omega]^{l^{(t)}} \to \mathcal{P}(\omega^{(k+\sum_{j=1}^{\bar{t}} l^{(j)})})$$

associated with R above. Now consider a pair of sequences $i_1 < \cdots < i_t$, $i_1' < \cdots < i_t'$, satisfying $i_1 = i_1'$ and for $1 < s \leqslant t$, satisfying $i_1 = i_1'$ and for $1 < s \leqslant t$, $i_s \equiv i_s' \pmod n$, families of sequences $y_1 \in [X_{i_1}']^{l^{(1)}}, \ldots, y_t \in [X_{i_t}']^{l^{(t)}}$, $y_1' \in [X_{i'}']^{l^{(1)}}, \ldots, y_t' \in [X_{i'}']^{l^{(t)}}$ and $y \in X_{i_1}'$ with $y < \min\{y_{10}, y_{10}'\}$. Then for any $z < y$, we have

$$\begin{aligned} R(z; y_1, \ldots, y_t) &\Leftrightarrow (z; y_1, \ldots, y_{\bar{t}}) \in F(y_t). \\ &\Leftrightarrow R^{(q)}(z; y_1, \ldots, y_{\bar{t}}) \\ &\Leftrightarrow R^{(q)}(z; y_1', \ldots, y_{\bar{t}}') \in F(y_t') \\ &\Leftrightarrow R(z; y_1', \ldots, y_t'), \end{aligned}$$

where $q \equiv i_t \equiv i_t' \pmod n$ and $1 \leqslant q \leqslant n$. From the claim it follows at once that the sequence $X_1', X_2', \ldots$ satisfies the requirements (1)–(3) of the lemma. □

5. Inductive towers

In this section we will construct a tower

$$\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$$

of models of Peano arithmetic with certain special properties. These properties will make it possible to use the models $\mathscr{A}_1, \mathscr{A}_2, \ldots$ to build Kripke models of epistemic arithmetic over infinite trees.

Let $L^\#$ be the language of arithmetic extended by adding a countable collection $P_0, P_1, \ldots$ of new unary predicate symbols. If

$$A(z_1, \ldots, z_n; P_0, \ldots, P_m)$$

is a formula of $L^\#$, $\mathscr{A}_0, \ldots, \mathscr{A}_m$ are models of arithmetic and $a_1, \ldots, a_n \in \bigcup_{i=1}^m \mathscr{A}_i$, then the notion of the many sorted structure $(\mathscr{A}_0, \ldots, \mathscr{A}_m)$ *satisfying* A at $a_1, \ldots, a_n$, denoted by

$$(\mathscr{A}_0, \ldots, \mathscr{A}_m) \models A[\boldsymbol{a}]$$

is defined in the obvious way: interpreting P_i by $\mathscr{A}_i$.

5.1. Definition. The collection of *regular formulas* of $L^\#$ is defined inductively by the following clauses:

(1) Each atomic formula of L is regular.

(2) If A and B are regular, then $(A \rightarrow B)$ is regular.

(3) If A is regular and if $i \leqslant j$ for each P_j occurring in A, then $\forall x\, (P_i(x) \rightarrow A)$ and $\exists x\, (P_i(x) \wedge A)$ are regular.

5.2. Definition. Let $C_1, \ldots, C_n$ be sentences of arithmetic. An *inductive tower for* $C_1, \ldots, C_n$ is a tower

$$\mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \cdots$$

of models of (PA) satisfying the following conditions:

(1) For each k, $1 \leqslant k \leqslant n$, and each $i \equiv k \pmod n$, $\mathscr{A}_i \models C_k$.

(2) For each regular formula $A(\boldsymbol{x}, \boldsymbol{z}; P_0, \ldots, P_m)$ and each sequence $i_0 < \cdots < i_m$ of positive integers

$$\begin{aligned}(\mathscr{A}_{i_0}, \ldots, \mathscr{A}_{i_m}) \models \forall \boldsymbol{z}\, [P_0(z_1) \wedge \cdots \wedge P_0(z_k) \rightarrow \\ (A(0) \wedge \forall x\, (P_0(x) \wedge A(x) \rightarrow A(x')) \rightarrow \forall x\, (P_0(x) \rightarrow A(x)))].\end{aligned}$$

(3) For each regular formula $A(\boldsymbol{z}; P_1, \ldots, P_s)$, each pair of sequences $i_1 < \cdots < i_s$ and $i_1' < \cdots < i_s'$, satisfying $i_1 = i_1'$ and for $1 < q \leqslant s$, $i_q \equiv i_q' \pmod n$, if $\boldsymbol{a} \in \mathscr{A}_{i_1}$, then $(\mathscr{A}_{i_1}, \ldots, \mathscr{A}_{i_s}) \models A(\boldsymbol{a})$ iff $(\mathscr{A}_{i_1'}, \ldots, \mathscr{A}_{i_s'}) \models A(\boldsymbol{a})$.

The main objective of this section is to show that if each C_k, $1 \leqslant k \leqslant n$, is consistent with (PA) + all true Π_1^0-sentences, then there exists an inductive tower for $C_1, \ldots, C_n$. For this we use the combinatorial principle of Section 4 to show the consistency of a Paris–Harrington type theory allowing ω-cuts.

Let L^b be the language of arithmetic extended by adding new constant symbols $\{c_{ij}\}_{1 \leqslant i \leqslant \omega, 1 \leqslant j < \omega}$. We will need the following notation. Let $j_1 < \cdots < j_r$ be a sequence of positive integers. Then $c_i(\boldsymbol{j})$ denotes the sequence of constants

$c_{ij_1}, \ldots, c_{ij_r}$. Let $A(z)$ be a formula of arithmetic with prenex normal form

$$\exists x_1 \cdots \forall x_r A_0(z; x),$$

where A_0 is quantifier-free. Then

$$A^*(z, y) \equiv \exists x_1 < y_1 \cdots \forall x_r < y_r A_0(z; x).$$

5.3. Definition. Let $C_1, \ldots, C_n$ be sentences of arithmetic. The theory $T(C_1, \ldots, C_n)$ has the following axioms

(1) The standard axioms for 0, ( )′.

(2) The recursive defining equations for $+$, $\cdot$ and $<$.

(3) Induction for all formulas of L^b.

(4) For each pair i, j the axiom $c_{ij}^2 < c_{i(j+1)}$.

(5) For all $i < i'$ and all j, j' the axiom $c_{ij} < c_{i'j'}$.

(6) For each bounded formula $B(z; y_1, \ldots, y_s)$, each pair of sequences $i_1 < \cdots < i_s$ and $i'_1 < \cdots < i'_s$ satisfying $i_1 = i'_1$ and for $1 \leq q \leq s$, $i_q \equiv i'_q \pmod n$ each family of sequences $j_1, \ldots, j_s$, $j'_1, \ldots, j'_s$ satisfying j_q, j'_q have the same length as y_q, for $1 \leq q \leq s$, and each $j < \min\{j_{10}, j'_{10}\}$, the axiom

$$\forall z < c_{i,j} [B(z; c_{i_1}(j_1), \ldots, c_{i_s}(j_s)) \Leftrightarrow B(z, c_{j'_1}(j'_1), \ldots, c_{i'_s}(j'_s))].$$

(7) For each k, $1 \leq k \leq n$, each $i \equiv k \pmod n$ and each sequence j of the proper length, the axiom $C_k^*(c_i(j))$.

5.4. Lemma. *Suppose C is a sentence of arithmetic consistent with* (PA) + *all true Π_1^0-sentences. For all N, t, e, r and h there exists a finite set X of natural numbers satisfying the following*:

(1) $X \underset{t}{\rightarrow} (N)_r^e$.

(2) *For each $x \in X$, $x > h$.*

(3) *For $x, y \in X$ if $x < y$, then $x^2 < y$.*

(4) *For any sequence $y_1 < \cdots < y_k$ from X, of the proper length, $C^*(y)$ is valid.*

Proof. We prove the lemma in the case when C is Π_3^0, which is the case we need; however, the proof of the general case can be obtained by very little modification. Suppose $C \equiv \forall x_1 \exists x_2 \forall x_3 B(x)$, where B is quantifier free.

Construct $D(x)$ a formula of arithmetic so that the following are theorems of (PA):

(1) $\forall x (D(x) \rightarrow x > \bar{h})$,

(2) $\forall xy (D(x) \wedge D(y) \wedge x < y \rightarrow x^2 < y)$,

(3) $\forall x \exists y > x D(y)$,

Consider the partition relation defined by

$$F(y_1, y_2, y_3) = \begin{cases} 0 & \text{if } \forall x_1 < y_1 \exists x_2 < y_2 \forall x_3 < y_3 B(x), \\ 1 & \text{otherwise.} \end{cases}$$

By the formalized version of Ramsey's Theorem, let $E(x)$ be a formula of arithmetic such that the following are theorems of (PA):

(1) $\forall x\,(E(x)\rightarrow D(x))$,
(2) $\forall x\,\exists y>x\,E(y)$,
(3) $\forall y_1, y_2, y_3, y_1', y_2', y_3'\,[y_1<y_2<y_3 \wedge y_1'<y_2'<y_3' \wedge$

$$\bigwedge_{i=1}^{3} (E(y_i) \wedge E(y_i'))\rightarrow F(y_1, y_2, y_3) = F(y_1', y_2', y_3')].$$

Now let $\mathscr{A}$ be a model of (PA) + all true Π_1^0-sentences + C. One easily shows that

$$\mathscr{A} \models \forall y_1y_2y_3\,(y_1<y_2<y_3 \wedge \bigwedge_{i=1}^{3} E(y_i)\rightarrow C^*(\mathbf{y})).$$

Applying Corollary 4.5, we obtain

$$(\mathrm{PA}) \vdash \exists X\,(\text{“}X \text{ is a finite set”} \wedge \forall x \in X\,E(x) \wedge X \underset{\bar{t}}{\rightarrow} (N)_r^e)).$$

From our choice of $E(x)$, it follows that

$$\begin{aligned}\mathscr{A} \models \exists X\,(&\text{“}X \text{ is a finite set”} \wedge X \underset{\bar{t}}{\rightarrow} (N)_r^e\\ &\wedge \forall x \in X\,(x>\bar{h}) \wedge \forall x, y \in X\,(x<y\rightarrow x^2<y)\\ &\wedge \forall y_1, y_2, y_3 \in X\,(y_1<y_2<y_3\rightarrow C^*(\mathbf{y}))).\end{aligned}$$

But this sentence is Σ_1^0, so it must also be valid in the standard model. □

5.5. Theorem. *Let $C_1, \ldots, C_n$ be sentences of arithmetic such that each C_q, $1 \leq q \leq n$, is consistent with* (PA) + *all true Π_1^0-sentences. Then the theory $T(C_1, \ldots, C_n)$ is consistent.*

Proof. Let S be a finite subset of $t(C_1, \ldots, C_n)$. We will expand the standard model $\mathcal{N}$ of arithmetic by giving interpretations to the new constant symbols appearing in S. Axioms 5.3(1)–5.3(3) will then be trivially valid.

For each formula $B(z; y_1, \ldots, y_t)$ of arithmetic, let R_B be the relation of type $(k; l^{(1)}, \ldots, l^{(t)})$ that B defines in the standard model. Let $\mathscr{R}$ be the family of relations

$\{R_B$; the instance of axiom 5.3(6) corresponding to B appears in $S\}$.

Suppose the rank of $\mathscr{R}$ is (t, N). Let $M>N$ be large enough so that for each constant symbol c_{ij} appearing in S, $j<M$.

Using Lemma 5.4, we can construct a sequence of finite subsets $X_1, X_2, \ldots$ of w satisfying the conditions:

(1) $X_j \underset{t+1}{\longrightarrow} (M)_{rj}^M$.
(2) If $i<i'$, $x \in X_i$ and $y \in X_{i'}$, then $x<y$.
(3) For x, $y \in X_i$ if $x<y$, then $x^2<y$.

(4) For each q, $1 \leq q \leq n$, each $i \equiv q \pmod n$ and each increasing sequence $\boldsymbol{x}$ from X_i, of the proper length, $C_q^*(\boldsymbol{x})$ is valid.

Here again, $r_0 = 0$ and $r_{i+1} = 2^{(\sup X_i)^N}$.

Now we apply Lemma 4.8 to obtain a sequence $X_1', X_2', \ldots$ such that $X_i' \subseteq X_{\pi(i)}$, where π is increasing and $\pi(i) \equiv i \pmod n$, $X_1', X_2', \ldots$ is n-homogeneous for $\mathscr{R}$ and $\|X_i'\| > M$ for all i. Finally, interprete $c_i, c_{i2}, \ldots, c_{iM}$ as an increasing sequence from X_i'. Conditions (2)–(4) above guarantee that axioms 5.3(4), 5.3(5) and 5.3(7) will be valid. Also, the homogeneity of $X_1', X_2', \ldots$ for $\mathscr{R}$ clearly suffices to validate the axioms of 5.3(6) appearing in S. □

For the remainder of this section we will assume that $C_1, \ldots, C_n$ are sentences of arithmetic each of which is consistent with (PA) + all true Π_1^0-sentences. By Theorem 5.5 the theory $T(C_1, \ldots, C_n)$ is consistent. Let $\mathscr{A}_\omega$ be a model of $T(C_1, \ldots, C_n)$. For each i let A_i be the initial segment of $\mathscr{A}_\omega$ consisting of those $a < c_{ij}$ for some j. By axiom 5.3(4), A_i is closed under $(\ )'$, $+$ and $\cdot$. Let $\mathscr{A}_i = \langle A_i, 0, (\ )', +, \cdot, \langle\ \rangle$ be the submodel of $\mathscr{A}_\omega$ corresponding to A_i. By 5.3(5), we obtain a tower of models

$$\mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \cdots \mathscr{A}_\omega.$$

Any regular formula of $L^\#$ $A(z; P_1, \ldots, P_t)$ is logically equivalent to a relativized prenex normal form

$$(\alpha) \qquad (Q_{10})_{P_1} x_{10} \cdots (Q_{1r_1-1})_{P_1} x_{1r_1-1} \cdots (Q_{t0})_{P_t} x_{t0} \cdots (Q_{tr_t-1})_{P_t} x_{tr_t-1}\, B(z; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_t)$$

where for $1 \leq i \leq t$, $0 \leq j < r_i$, Q_{ij} is either $\forall$ or $\exists$ and B is quantifier-free.

5.6. Definition. Let $A(z; P_1, \ldots, P_t)$ be a regular formula of $L^\#$ with prenex normal form (α) above. Then $\hat{\phi}(z; \boldsymbol{y}_1, \ldots, \boldsymbol{y}_t)$ is the arithmetic formula

$$Q_{10} x_{10} < y_{10} \cdots Q_{1r_1-1} x_{1r_1-1} < y_{1r_1-1} \cdots Q_{t0} x_{t0} < y_{t0} \cdots Q_{tr_t-1} x_{tr_t-1} < y_{tr_t-1}\, B(z; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_t).$$

5.7. Lemma. *Let $A(z; P_1, \ldots, P_t)$ be a regular formula of $L^\#$. For any sequence $i_1 < \cdots < i_t$, any sequences $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_t$ of the proper length and any $k < k_{10}$ we have*

$$(\mathscr{A}_{i_1}, \ldots, \mathscr{A}_{i_t}) \vDash A[\boldsymbol{a}] \quad \textit{if } \mathscr{A}_w \vDash \hat{A}(\boldsymbol{a}, c_{i_1}(\boldsymbol{k}_1), \ldots, c_{i_t}(\boldsymbol{k}_t))$$

for all $\boldsymbol{a} \in \mathscr{A}_{i_1}$ with $\boldsymbol{a} < c_{i_1k}$.

Proof. A simple induction on the complexity of A, using axiom 5.3(6). □

5.8. Lemma. *Let $a(z; P_1, \ldots, P_t)$ be a regular formula of $L^\#$ and let $i_1 < \cdots < i_t$ and $i_1' < \cdots < i_t'$ be sequences of positive integers satisfying $i_1 = i_1'$ and for*

$1 \leqslant q \leqslant t$, $i_q \equiv i'_q \pmod n$. *Then for any* $\boldsymbol{a} \in \mathcal{A}_{i_1}$, *we have*

$$(\mathcal{A}_{i_1}, \ldots, \mathcal{A}_{i_t}) \models A(\boldsymbol{a}) \quad \textit{iff} \quad (\mathcal{A}_{i'_1}, \ldots, \mathcal{A}_{i'_t}) \models A[\boldsymbol{a}].$$

Proof. This follows at once from Lemma 5.7 and axiom 5.3(6). □

5.9. Lemma. *Let* $A(x, z; P_0, \ldots, P_t)$ *be a regular formula of* $L^{\#}$. *For any sequence* $i_0 < \cdots < j_t$ *of positive integers*

$$(\mathcal{A}_{i_0}, \ldots, \mathcal{A}_{i_t}) \models \forall z\, [P_0(z_i) \wedge \cdots \wedge P_0(Z_k) \rightarrow (A(0) \wedge \forall x\, (P_0(x) \wedge A(x) \\ \rightarrow A(x')) \rightarrow \forall x\, (P_0(x) \rightarrow A(x)))].$$

Proof. Let $\boldsymbol{a} \in \mathcal{A}_{i_0}$ and assume $\mathcal{A} \models A(0, \boldsymbol{a}) \wedge \forall x\, (P_0(x) \wedge A(x) \rightarrow A(x'))$. Let k, $\boldsymbol{k}_0$, $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_t$ be sequences of the appropriate length $j < k$ be such that $\boldsymbol{a} < c_{i_0 j}$. Then

$$\mathcal{A}_\omega \models \hat{A}(0, \boldsymbol{a}; c_{i_0}(\boldsymbol{k}_0), \ldots, c_{i_t}(\boldsymbol{k}_t)) \wedge \\ \forall x < c_{i_0 k}[\hat{A}(x, \boldsymbol{a}; c_{i_0}(\boldsymbol{k}_0), \ldots, c_{i_t}(\boldsymbol{k}_t)) \rightarrow \hat{A}(x', \boldsymbol{a}; c_{i_0}(\boldsymbol{k}_0), \ldots, C_{i_t}(\boldsymbol{k}_t))].$$

From this we easily get

$$\mathcal{A}_\omega \models \forall x < c_{i_0 k}\, \hat{A}(x, \boldsymbol{a}; c_{i_0}(\boldsymbol{k}_0), \ldots, c_{i_t}(\boldsymbol{k}_t)).$$

Hence

$$\mathcal{A} \models \forall x\, (P_0(x) \rightarrow A(x, \boldsymbol{a})). \quad \square$$

5.10. Lemma. *For each i,* $\mathcal{A}_i$ *is a model of* (PA) *and if* $q \equiv i \pmod n$, $1 \leqslant q \leqslant n$, *then* $\mathcal{A}_i \models C_q$.

Proof. It follows at once from Lemma 5.9 and the fact that $\mathcal{A}_i$ is a submodel of $\mathcal{A}_\omega$ that $\mathcal{A}_i$ is a model of (PA). Also, since

$$\mathcal{A}_\omega = C_q^*(c_i(\boldsymbol{j}))$$

for each increasing sequence $\boldsymbol{j}$ of the proper length, it follows from Lemma 5.7 that $\mathcal{A}_i \models C_q$. □

Combining Lemmas 5.8–5.10, we obtain

5.11. Theorem. *Let* $C_1, \ldots, C_n$ *be sentences of arithmetic such that each* C_q, $1 \leqslant q \leqslant n$, *is consistent with* (PA) + *all* Π_1^0*-sentences. Then there exists an inductive tower for* $C_1, \ldots, C_n$.

6. Maximality for epistemic logic

We are finally ready to prove our main result asserting the maximality of epistemic propositional logic for epistemic arithmetic.

Suppose $A(p_1, \ldots, p_m)$ is a formula of (EP) such that

$$\nvdash_{\mathrm{EP}} A.$$

Choose $B_1, \ldots, B_m$ Π_2^0-sentences of arithmetic completely independent over (PA) + all true Π_1^0-sentences. Let $C_1, \ldots, C_n$ be an enumeration of the sentences of the form

$$\bigwedge_{i \in I} B_i \wedge \bigwedge_{i \notin I} \neg B_i,$$

for $I \subseteq \{1, \ldots, m\}$. Each sentence C_i is consistent with (PA) + all true Π_1^0-sentences. By Theorem 5.10, there is an inductive tower $\mathscr{A}_1, \mathscr{A}_2, \ldots$ for $C_1, \ldots, C_n$.

By the completeness theorem for (EP), let $\mathscr{K} = \langle T, \leqslant, \Vdash \rangle$ be a Kripke model of finite type such that

$$\nVdash^{\mathscr{K}}_{\langle\,\rangle} A.$$

Suppose T has M types, and let $\tau(\alpha)$ denote the type of α, for $\alpha \in T$.

To each node α of T, assign a positive integer $\#(\alpha)$ inductively by the clause

$$\#(\alpha) = \begin{cases} \text{the least } l \text{ such that } \forall \beta < \alpha\, [\#(\beta) < l \text{ and} \\ \text{for } i = 1, 2, \ldots, m\ (\mathscr{A}_l \models B_i \iff \Vdash_\alpha P_i)]. \end{cases}$$

Since $i < j$ implies $\mathscr{A}_i \subseteq \mathscr{A}_j$, the assignment $\alpha \to \mathscr{A}_{\#(\alpha)}$ defines a quantificational Kripke model. Denote this model by $\bar{\mathscr{K}}$.

6.1. Lemma. *For each formula $D(P_1, \ldots, P_m)$ of* (EP) *and each node α of T,*

$$\Vdash^{\mathscr{L}}_{\alpha} D \quad \textit{iff} \quad \Vdash^{\bar{\mathscr{K}}}_{\alpha} D(B_1, \ldots, B_m).$$

Therefore to show $\nvdash_{(\mathrm{EA})} A(B_1, \ldots, B_m)$ it will suffice to show $\bar{\mathscr{K}}$ satisfies the axioms of (EA). Since each $\mathscr{A}_i$ is a model of (PA), $\bar{\mathscr{K}}$ satisfies all the axioms except possibly the axiom of induction. We will establish the validity of induction by showing that the forcing relation of $\bar{\mathscr{K}}$ is 'definable' in the appropriate many sorted structure.

From the definition of $\#(\alpha)$ above, it is clear that for $\alpha, \beta \in T$, if $\tau(\alpha) = \tau(\beta)$, then $\#(\alpha) \equiv \#(\beta) \pmod n$.

6.2. Definition. *For each formula $D(z)$ of* (EA) *and each type τ we assign a natural number t_D^τ and a regular formula $D^\tau(z; P_0, \ldots, P_{t_D})$ by induction on the complexity of D as follows:*

(1) *If D is atomic, $t_D^\tau = 0$ and $D^\tau \equiv D$.*

(2) *If $D \equiv (E \to F)$, then $t_D^\tau = \max\{t_E^\tau, t_F^\tau\}$ and $D^\tau \equiv (E^\tau \to F^\tau)$.*

(3) *If $D \equiv \forall x\, E$, then $t_D^\tau = t_E^\tau$ and $D^\tau \equiv \forall x\, (P_0(x) \to E^\tau)$.*

(4) *If $D \equiv \Box E$, let $\tau_1, \ldots, \tau_k$ be all the types that occur above α for $\alpha \in T$, with $\tau(\alpha) = \tau$, and let r_j, for $1 \leqslant j \leqslant k$, be the least $r > 0$ such that $\#(\alpha) +$*

$r \equiv \#(\alpha_j) \pmod n$ *for any* α *with* $\tau(\alpha) = \tau$ *and* α_j *with* $\tau(\alpha_j) = \tau_j$. *Then*

$$t_D^\tau = \max\{t_E^\tau, r_1 + t_E^{\tau_1}, \ldots, r_k + t_E^{\tau_k}\}$$

and

$$D^\tau \equiv E^\tau \wedge \bigwedge_{j=1}^{k} E^{\tau_j}(z; P_{r_j}, \ldots, P_{r_j + t_E^{\tau_j}}).$$

6.3. Lemma. *Let* $D(z)$ *be a formula of* (EA) *and let* $\alpha \in T$ *with* $\tau = \tau(\alpha)$. *Then for all* $a_1, \ldots, a_l \in \mathscr{A}_\alpha$,

$$\Vdash_\alpha D[\boldsymbol{a}] \quad \textit{iff} \quad (\mathscr{A}_{\#(\alpha)}, \ldots, \mathscr{A}_{\#(\alpha)+t_D^\tau}) \models D^\tau[\boldsymbol{a}].$$

Proof. We argue by induction on the complexity of D. The only case that is not quite clear is when $D \equiv \Box E$. With the same notation as in clause (4) of Definition 6.2, we have $\Vdash_\alpha D \quad \Leftrightarrow \quad \Vdash_\alpha E$ and

$$\bigwedge_{j=1}^{k} \forall \beta > \alpha\, [\tau(\beta) = \tau_j \Rightarrow \Vdash_\beta E]$$

$$\Leftrightarrow \quad (\mathscr{A}_{\#(\alpha)}, \ldots, \mathscr{A}_{\#(\alpha)+t_E^\tau}) \models E^\tau(P_0, \ldots, P_{t_E^\tau})$$

and

$$\bigwedge_{j=1}^{k} \forall \beta > \alpha\, [\tau(\beta) = \tau_j \Rightarrow (\mathscr{A}_{\#(\beta)}, \ldots, \mathscr{A}_{\#(\beta)+t_E^\tau}) \models E^{\tau_j}(P_0, \ldots, P_{t_E^{\tau_j}})].$$

By condition (3) in the definition of an inductive tower, for $\beta > \alpha$ with $\tau(\beta) = \tau_j$,

$$(\mathscr{A}_{\#(\beta)}, \ldots, \mathscr{A}_{\#(\beta)+t_E^{\tau_j}}) \models E^{\tau_j}(P_0, \ldots, P_{t_E^{\tau_j}})$$
$$\text{iff} \quad (\mathscr{A}_{\#(\alpha)+r_j}, \ldots, \mathscr{A}_{\#(\alpha)+r_j+t_E^{\tau_j}}) \models E^{\tau_j}(P_0, \ldots, P_{t_E^{\tau_j}}).$$

It follows that

$$\Vdash_\alpha D \quad \Leftrightarrow \quad (\mathscr{A}_{\#(\alpha)}, \ldots, \mathscr{A}_{\#(\alpha)+t_E^\tau}) \models E^\tau(P_0, \ldots, P_{t_E^\tau}).$$

and

$$\bigwedge_{j=1}^{k} (\mathscr{A}_{\#(\alpha)+r_j}, \ldots, \mathscr{A}_{\#(\alpha)+r_j+t_E^{\tau_j}}) \Vdash E^{\tau_j}(P_0, \ldots, P_{t_E^{\tau_j}})$$

$$\Leftrightarrow \quad (\mathscr{A}_{\#(\alpha)}, \ldots, \mathscr{A}_{\#(\alpha)+t_D^\tau}) \models D^\tau(P_0, \ldots, P_{t_D^\tau}). \qquad \Box$$

6.4. Lemma. *The axiom scheme of induction is valid in* $\tilde{\mathscr{K}}$.

Proof. Consider the instance of the induction axiom

$$D(0) \wedge \forall x\, (D(x) \rightarrow D(x')) \rightarrow \forall x\, D(x).$$

If we denote this formula by E, then

$$E^\tau \equiv D^\tau(0) \wedge \forall x\, (P_0(x) \rightarrow (D^\tau(x) \rightarrow D^\tau(x'))) \rightarrow \forall x\, (P_0(x) \rightarrow D^\tau(x)).$$

Hence the validity of E in $\bar{\mathcal{K}}$ now follows from Lemma 6.3 and caluse (2) of Definition 5.2. □

It follows that $\bar{\mathcal{K}}$ satisfies all the axioms of (EA). Consequently, by Theorem 2.13, $\bar{\mathcal{K}}$ satisfies all theorems of (EA). Also by Lemma 6.1, $\bar{\mathcal{K}}$ does not satisfy $A(B_1, \ldots, B_m)$.

6.4. Theorem. *Suppose* $A(P_1, \ldots, P_m)$ *is a formula of* (EP) *such that*

$$\nvdash_{\mathrm{EP}} A.$$

Then there are Π_2^0*-sentences* $B_1, \ldots, B_m$ *of arithmetic such that*

$$\nvdash_{\mathrm{EA}} A(B_1, \ldots, B_m).$$

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